Engineering Notes

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Non-Newtonian Stagnation Flow with Mass Injection

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The mathematical problem of external boundary-layer flow of a non-Newtonian fluid was first investigated by Schowalter[1] and later by Acrivos, Shah, and Petersen.[2] Both papers have established the conditions for the existence of similarity solutions for pseudoplastic, power-law-type fluids, and the latter has also included numerical results for the solution of flow over a flat plate. The extensions of the general problem to less restrictive classes of non-Newtonian, viscoelastic fluids were presented by Wells,[3] Lee and Ames,[4] and White and Metzner,[5] depending on the different models for relating stress and rate of deformation tensors.

The solutions, presented in the aforementioned investigations, were obtained under the condition of impermeable walls. A recent study by Thompson and Snyder[6] included the effect of wall mass injection for the flat-plate problem, and the numerical results indicated the feasibility of further drag reduction to the non-Newtonian characteristics of the fluid.

This Note presents the results for the stagnation flow problem of a non-Newtonian, power-law type fluid with wall mass injection consistent with the conditions for the existence of similarity solutions for the mathematical system.

The governing equations for steady, two-dimensional flow of an incompressible fluid, under Prandtl first-order boundary-layer approximations, are

$$(\partial u/\partial x) + (\partial v/\partial y) = 0 \tag{1}$$

$$u(\partial u/\partial x) + v(\partial u/\partial y) = U(dU/dx) + (\partial/\partial y)(\tau_{xy}) \tag{2}$$

where x and y are the space coordinates along and transverse to the surface, respectively, with the associated velocity components u and v. The inviscid, external velocity distribution along x, $U(x)$, is superimposed on the viscous region as a consequence of the classical approximation of no pressure variation across the boundary layer.

For a non-Newtonian, Ostwald-de Waele, power-law type fluid, the stress component τ_{xy} is given as

$$\tau_{xy} = \nu(\partial u/\partial y)^N \tag{3}$$

where ν and N are constant parameters for the modified stress rate of deformation relation.

For problems including wall mass injection, the mathematical system, Eqs. (1–3), is to be solved subject to the boundary conditions

$$u(x,0) = 0 \qquad v(x,0) = V_w(x)$$
$$\lim_{y\to\infty} u(x,y) = U(x) \tag{4}$$

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where $V_w(x)$ represents the injection velocity distribution on the surface.

Taking a characteristic length L along the surface, the modified form of the dimensionless Reynolds number R_e is defined in terms of the coefficients of viscosity term ν and N, and the magnitude of the freestream velocity U_∞ as

$$R_e = U^{(2-N)}{}_\infty L^N/\nu \tag{5}$$

Defining the dimensionless quantities

$$\bar{x} = x/L \qquad \bar{u} = u/U_\infty \qquad \bar{U} = U/U_\infty$$
$$\bar{y} = (y/L)R^{1/(1+N)}{}_e \qquad \bar{v} = (v/U_\infty)R^{1/(1+N)}{}_e \tag{6}$$

and the dimensionless stream function $\bar{\Psi}(\bar{x},\bar{y})$, which satisfies the continuity equation (1), identically,

$$\bar{u} = \partial\bar{\Psi}/\partial\bar{y} \qquad \bar{v} = -\partial\bar{\Psi}/\partial\bar{x} \tag{7}$$

and substituting (6) and (7) in (2), the governing dimensionless equation becomes

$$\frac{\partial\bar{\Psi}}{\partial\bar{y}}\frac{\partial^2\bar{\Psi}}{\partial\bar{x}\partial\bar{y}} - \frac{\partial\bar{\Psi}}{\partial\bar{x}}\frac{\partial^2\bar{\Psi}}{\partial\bar{y}^2} = \bar{U}\frac{d\bar{U}}{d\bar{x}} + \frac{\partial}{\partial\bar{y}}\left(\frac{\partial^2\bar{\Psi}}{\partial\bar{y}^2}\right)^N \tag{8}$$

Applying the general similarity via the one-parameter group theory by Birkhoff,[7] and requiring the condition of constant conformally invariant transformation properties, the similarity variables for the system become

$$\eta = \bar{y}\{\bar{x}^{[(N-2)p+1]/(N+1)}\}^{-1}$$
$$f(\eta) = \bar{\Psi}(\bar{x},\bar{y})\{\bar{x}^{[(2N-1)p+1]/(N+1)}\} \tag{9}$$

where p is defined according to the condition of the existence of similarity solutions for the freestream velocity $U(\bar{x})$ as

$$U(x) = U_\infty\bar{U}(\bar{x}) = U_\infty\bar{x}^p = U_\infty(x/L)^p \tag{10}$$

The transformed dimensionless velocity components, in terms of $f(\eta)$, become

$$\bar{u} = \bar{x}^p f'$$
$$\bar{v} = \bar{x}^{[(2N-1)p-N]/(N+1)}\left[\frac{(N-2)p+1}{(N+1)}\eta f' - \frac{(2N-1)p+1}{(N+1)}f\right] \tag{11}$$

and the boundary conditions (4) transform to

$$f'(0) = 0 \qquad f(0) = -V_w^*$$
$$\lim_{\eta\to\infty} f'(\eta) = 1 \tag{12}$$

where V_w^* is a constant parameter of the wall mass injection velocity,

$$V_w(x) = \{[(2N-1)p+1]/(N+1)\} \times V_w^*(x/L)^{[(2N-1)p-N]/(N+1)} \tag{13}$$

which represents the required restrictive variation of the injection velocity on the surface. The transformed similar

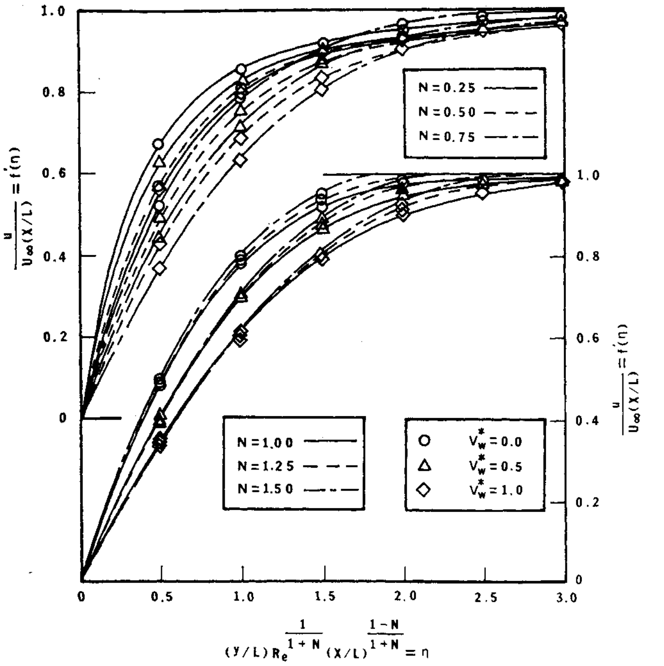

Fig. 1 Dimensionless velocity profiles.

differential equation for $f(\eta)$ becomes

$$f''' + \left[\frac{(2N-1)p+1}{N(N+1)}\right]f(f'')^{2-N} + \frac{p}{N}(f'')^{1-N}[1-(f')^2] = 0 \quad (14)$$

For stagnation flow conditions, the external, inviscid, freestream velocity distribution is

$$U(x) = U_\infty(x/L) = U_\infty\bar{x} \qquad p = 1 \quad (15)$$

Hence, the mathematical system, for the stagnation flow of a non-Newtonian fluid, with wall mass injection, reduces to

$$f''' + \frac{2}{(N+1)}f(f'')^{2-N} + \frac{1}{N}(f'')^{1-N}[1-(f')^2] = 0 \quad (16)$$

$$f(0) = -V_w^* \qquad f'(0) = 0$$
$$\lim_{\eta\to\infty} f'(\eta) = 1 \quad (17)$$

The ordinary differential equation (16) was integrated by the fourth-order Runge-Kutta method with Gill coefficients by employing a modified version of the scheme developed by Nachtsheim and Swigert[8] for the successive iterative guesses on the initial condition $f''(0)$.

The results of the similar, dimensionless velocity profiles, $f'(\eta)$, are presented in Fig. 1 for various values of the non-Newtonian power law index N and the wall mass injection parameter V_w^*. For $N < 1$, the figures indicate that the velocity profiles tend to reach freestream condition $f(\eta) = 1$ slower as N increases toward the Newtonian fluid conditions $N = 1$. Consequently, the boundary-layer thickness, where the viscous effects are dominant, increases with N. This result is to be expected since the exponent of the power law N is the controlling parameter of viscous effects. The maximum difference between $N = 0.25$ and $N = 0.75$ profiles reaches 30% at $\eta = 0.5$; hence, noting that the freestream conditions are reached at approximately $\eta = 4.0$, the changes in the profiles exist only in regions near to the wall. For $N \geq 1$ the same effect prevails, however, to a considerably lower degree. The maximum difference between $N = 1.0$ and $N = 1.5$ velocity profiles is approximately 4%.

For all values of N, the increase in wall injection parameter V_w^* has the strong effect of retarding the development of the boundary-layer velocity profiles, hence in a sense increasing the boundary-layer thickness. For $N < 1$, the difference between the velocity profiles without wall injection $V_w^* = 0$ and strong wall injection $V_w^* = 1.0$ varies between 15% for $N = 0.25$ and 50% for $N = 0.75$. For $N \geq 1$, the differences are approximately 20% for all N.

The dimensionless drag coefficient C_D is defined as

$$C_D = \frac{1}{LU_\infty^2}\int_0^L (\tau_{xy})_w dx = \frac{\nu}{LU_\infty^2}\int_0^L \left[\frac{\partial u}{\partial y}(x,0)\right]^N dx \quad (18)$$

Substituting the dimensionless similarity variables (9) in (18) and performing the straightforward integration for $p = 1$,

$$C_D = \frac{(N+1)}{(3N+1)}\frac{[f''(0)]^N}{R_e^{1/(1+N)}}$$
$$\frac{(3N+1)}{(N+1)}R_e^{1/(1+N)}C_D = [f''(0)]^N \quad (19)$$

The variations of the scaled drag coefficient $[f''(0)]^N$ with the wall mass injection parameter V_w^* are presented in Fig. 2, parametrically, for various values of N. The results indicate the strong influence of the wall injection and the exponent of the viscosity term N in reduction of the viscous drag. For low viscous effects $N = 0.25$ the drag coefficient is reduced by 15% by strong wall injection $V_w^* = 1.0$. However, for strong viscous effects $N = 1.50$ the strong wall mass injection $V_w^* = 1.0$ reduces the drag coefficient 52% from its value without mass injection. It is interesting to note that the variations of $[f''(0)]^N$ with V_w^* exhibit monotonically decreasing characteristics, with the negative slope of the curves increasing with the exponent of the viscous term N.

In summary, it is concluded that the viscous drag of the stagnation flow of a non-Newtonian fluid can be reduced considerably by the wall mass injection, and the degree of this reduction strongly depends on the exponent of the viscous term of the power-law model.

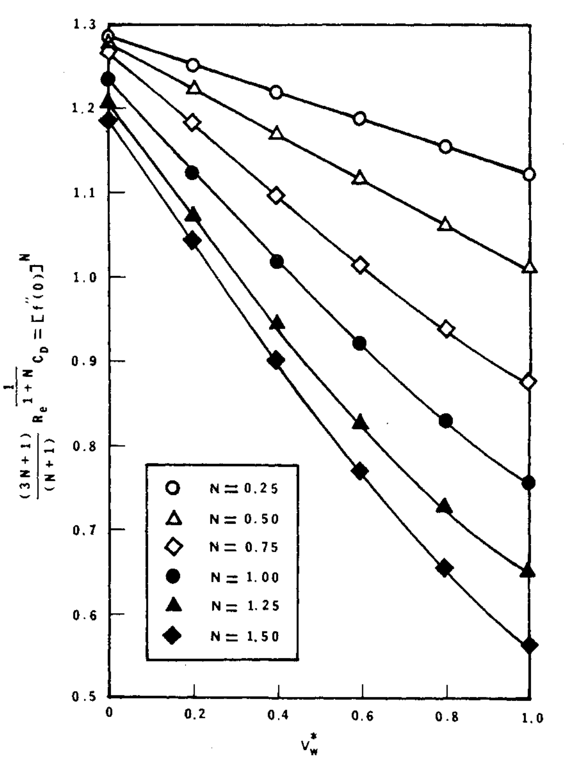

Fig. 2 Dimensionless, scaled drag coefficient.

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Axially Symmetric Incompressible Turbulent Wake Downstream of a Single Body

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Nomenclature

B	=	constant
b	=	half-width of wake
C	=	constant
C_D	=	drag coefficient
D	=	drag
d	=	diameter of body
l	=	mixing length
u	=	streamwise velocity component
u_1	=	$u_1 = u_\infty - u$
u_∞	=	freestream velocity
v	=	velocity component parallel to y coordinate
x	=	streamwise coordinate
y	=	coordinate perpendicular to x
β	=	constant
η	=	$\eta = y/b$
ρ	=	density of fluid
τ	=	shear stress

Introduction

The spread and stream velocity distribution of axially symmetric incompressible turbulent wake (Fig. 1) downstream of a body of revolution are determined. The uniformity of static pressure and similarity of the velocity profile are assumed, and, for the analysis, Prandtl's mixing length theory is applied. The obtained wake solution is an extension of the known two-dimensional analysis.

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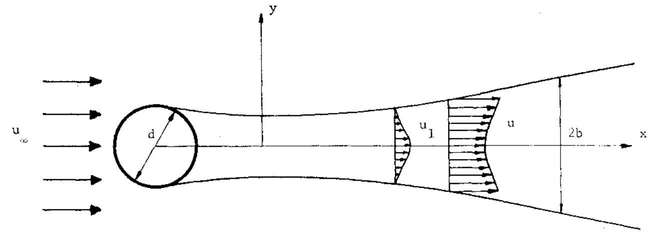

Fig. 1 Incompressible turbulent wake.

Wake Flow Analysis

The sketch shows the coordinate system and the velocity distribution. The governing equations are

Continuity:

$$[\partial(yu)/\partial x] + [\partial(yv)/\partial y] = 0$$

Momentum:

$$u(\partial u/\partial x) + v(\partial u/\partial y) = (1/y\rho)[\partial(y\tau)/\partial y]$$

In similar velocity profiles, the velocity difference u_1 defined by

$$u_1 = u_\infty - u$$

is small compared to u_∞. By substituting u_1 into these two equations and considering the order of magnitude, the momentum equation is reduced to

$$-u_\infty(\partial u_1/\partial x) = (\tau/y\rho) + (1/\rho)(\partial\tau/\partial y)$$

Now introducing Prandtl's mixing length l,

$$\tau = \rho l^2|\partial u/\partial y|\partial u/\partial y$$

we obtain

$$-u_\infty \frac{\partial u_1}{\partial x} = \frac{l^2}{y}\left(\frac{\partial u_1}{\partial y}\right)^2 + 2l^2 \frac{\partial u_1}{\partial y}\frac{\partial^2 u_1}{\partial y^2} \quad (1)$$

Since for an axially symmetric turbulent incompressible wake, b and u_1 are proportional to $x^{1/3}$ and $x^{-2/3}$, respectively,[1] it is assumed that b and u_1 are given by

$$b = B(C_D d^2 x)^{1/3} \qquad \text{and} \qquad \frac{u_1}{u_\infty} = \left(\frac{x^2}{C_D d^2}\right)^{-1/3} f(\eta)$$

The constant B is to be determined later.

Relating β by $l = \beta b$, Eq. (1) is reduced to

$$\tfrac{1}{3}(2f + \eta f') = 2(\beta^2/B)[f'f'' + (1/2\eta)f'^2]$$

The solution of this differential equation is

$$\tfrac{1}{3}\eta f = (\beta^2/B)f'^2 + C_1 \quad (2)$$

where $'$ denotes differentiation with respect to η.

From the boundary conditions at $\eta = 1, f = f' = 0$, it follows that $C_1 = 0$. Equation (2) then leads to

$$df/(f)^{1/2} = (B/3\beta^2)^{1/2}(\eta)^{1/2}d\eta$$

Integrating, we obtain

$$2(f)^{1/2} = \tfrac{2}{3}(B/3\beta^2)^{1/2}\eta^{3/2} + C_2$$

From the boundary condition at $\eta = 1, f = 0$, it follows that

$$C_2 = -\tfrac{2}{3}(B/3\beta^2)^{1/2}$$

Hence

$$f = \tfrac{1}{3}(B/9\beta^2)(1 - \eta^{3/2})^2$$

This result is similar to that of two-dimensional wake flow